Fourier Analysis II:

Some Examples of the Use of Fourier Analysis

A. Fourier Analysis of a Pure-Tone/Single Frequency Waveform

The simplest example of the use of Fourier analysis is that of determining the harmonic content of a pure tone, periodic waveform of a single frequency, f e.g. applied as the input stimulus to a system:

$$S_i(t) = A_i \cos(2\pi f t) = A_i \cos(\omega t)$$

where A_i is the *amplitude* of the input stimulus, $S_i(t)$ and $\omega = 2\pi f$ is the "angular" frequency, in units of *radians* per second. The *period*, τ of the waveform is $\tau = 1/f$, in units of seconds.

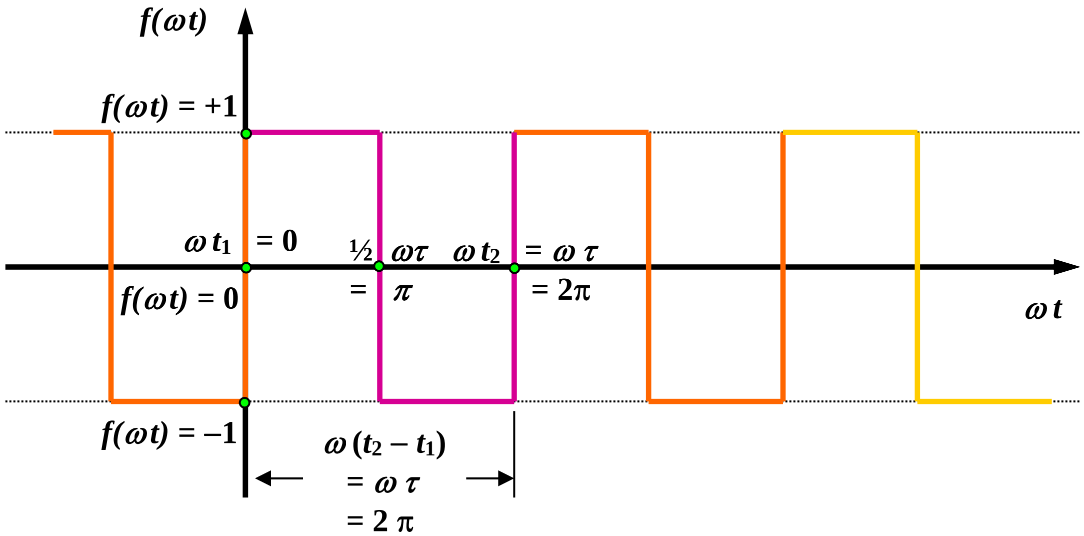
Then if the Fourier series representation of $S_i(t)$ is given by:

$$S_i(t) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos(n\omega t) + \sum_{n=1}^{n=\infty} b_n \sin(n\omega t) = A_i \cos(\omega t)$$

we see *by inspection* that for this equality to hold, the $n = 0$ coefficient, $a_o = 0$, and *all* of the $n > 0$ coefficients, a_n and b_n must also vanish, *except* for the a_1 coefficient, which must be $a_1 = A_i$. Note that these results can also be obtained by explicitly carrying out the inner products $< S_i(t), 1>$, $< S_i(t), \cos(n\omega t)>$ and $< S_i(t), \sin(n\omega t)>$, as defined above.

B. Fourier Analysis of a Periodic, Symmetrical Square Wave

A temporally-periodic, *bipolar* square wave of *unit* amplitude and 50% duty cycle is shown in the figure below:



Since this waveform repeats indefinitely, then, without any loss of generality we can *arbitrarily* choose (i.e. re-define) the starting time, t_1 of this waveform to be $t_1 = 0$ seconds. Thus the ending time, for one period of this waveform is $t_2 = \tau$ seconds. Then $\theta_1 = \omega t_1 = 0$, and $\theta_2 = \omega t_2 = \omega\tau = 2\pi f \tau = 2\pi/\tau * \tau = 2\pi$, since $f = 1/\tau$.

Mathematically, we define the square wave, for the one cycle as indicated in the figure above, as:

$$f(\theta) = f(\omega t) = +1 \quad \text{for} \quad 0 \le \theta < \pi$$

and:

$$f(\theta) = f(\omega t) = -1 \quad \text{for} \quad \pi \le \theta < 2\pi$$

This type of waveform is known as a *bipolar* square wave. It is positive, with unit amplitude for the first half of its cycle, and negative, with unit amplitude for the second half of its cycle. Thus, this waveform has a 50% duty cycle. Note also that this waveform has *odd* reflection symmetry, both about its θ-midpoint (i.e. its ωt-midpoint), $\theta = \omega t = \frac{1}{2}\omega\tau = \pi$, *and* reflection about its $f(\theta) = f(\omega t) = 0$ midpoint. The waveform is said to have *odd* symmetry if it changes sign upon reflection, and has *even* symmetry if it does not change sign upon reflection.

Note also that this waveform is an example of a function which ***is*** *piece-wise continuous*. The waveform has *discrete*, but *finite* "jumps" when $\theta = 0$, π and 2π. Mathematically, the *slopes* of this waveform at these θ-points, i.e. the θ-derivatives of $f(\theta)$ are formally infinite:

$$\partial f(\theta)/\partial\theta|_{\theta=0} = +\infty, \quad \partial f(\theta)/\partial\theta|_{\theta=\pi} = -\infty, \quad \text{and} \quad \partial f(\theta)/\partial\theta|_{\theta=2\pi} = +\infty.$$

This is OK - mathematically we can deal with this just fine!

Note further that this bipolar, 50% duty-cycle square wave has a "d.c. offset", or "time"-averaged value (averaged over one cycle) of $< f(\theta) > = 0$. Formally, the averaging of a periodic function over one of its cycles, which for the θ-variable, one cycle in theta is $\Delta\theta = \theta_2 - \theta_1 = 2\pi$, is mathematically defined as:

$$\langle f(\theta)\rangle = \frac{1}{\Delta\theta}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)d\theta = \frac{1}{2\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)d\theta$$

For the periodic bipolar, 50% duty-cycle square wave, the θ-averaging of this waveform over one θ-cycle is:

$$\langle f(\theta)\rangle = \frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)d\theta = \frac{1}{2\pi}\left[\int_{\theta=0}^{\theta=\pi} f(\theta)d\theta + \int_{\theta=\pi}^{\theta=2\pi} f(\theta)d\theta\right] = \frac{1}{2\pi}\left[\int_{\theta=0}^{\theta=\pi}(+1)d\theta + \int_{\theta=\pi}^{\theta=2\pi}(-1)d\theta\right]$$
$$= \frac{1}{2\pi}\left[\int_{\theta=0}^{\theta=\pi} d\theta - \int_{\theta=\pi}^{\theta=2\pi} d\theta\right] = \frac{1}{2\pi}\left[\theta\,|_{\theta=0}^{\theta=\pi} - \theta\,|_{\theta=\pi}^{\theta=2\pi}\right] = \frac{1}{2\pi}[\pi - \pi] = \frac{\pi}{2\pi}[1-1] = \frac{1}{2}[1-1] = 0$$

QED.

We now obtain the Fourier coefficients a_0, a_n and b_n by taking the following inner products:

$$a_0 = \frac{1}{\pi}\langle f(\theta),1\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)d\theta = \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi} f(\theta)d\theta + \int_{\theta=\pi}^{\theta=2\pi} f(\theta)d\theta\right]$$
$$= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi} d\theta - \int_{\theta=\pi}^{\theta=2\pi} d\theta\right] = \frac{1}{\pi}[\pi - \pi] = \frac{\pi}{\pi}[1-1] = \frac{1}{1}[1-1] = [1-1] = 0$$

Thus, if the reader compares the inner product for determining a_0 with that for obtaining the θ-averaged value of $f(\theta)$, i.e. $< f(\theta) >$, one sees that:

$$\langle f(\theta)\rangle = \frac{a_0}{2}$$

which is just what we expected!

Now, recall that here, since $\theta = \omega t$, the "generic" variable, $\theta_n = n\omega t = n\theta$. Thus:

$$a_n = \frac{1}{\pi}\langle f(\theta),\cos(\theta_n)\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)\cos(\theta_n)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)\cos(n\theta)d\theta$$
$$= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi} f(\theta)\cos(n\theta)d\theta + \int_{\theta=\pi}^{\theta=2\pi} f(\theta)\cos(n\theta)d\theta\right] = \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi}\cos(n\theta)d\theta - \int_{\theta=\pi}^{\theta=2\pi}\cos(n\theta)d\theta\right]$$

$$b_n = \frac{1}{\pi}\langle f(\theta),\sin(\theta_n)\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)\sin(\theta_n)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)\sin(n\theta)d\theta$$
$$= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi} f(\theta)\sin(n\theta)d\theta + \int_{\theta=\pi}^{\theta=2\pi} f(\theta)\sin(n\theta)d\theta\right] = \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi}\sin(n\theta)d\theta - \int_{\theta=\pi}^{\theta=2\pi}\sin(n\theta)d\theta\right]$$

Now the *indefinite* integrals:

$$\int \cos(n\theta)d\theta = +\frac{\sin(n\theta)}{n} \qquad \int \sin(n\theta)d\theta = -\frac{\cos(n\theta)}{n}$$

Thus, the Fourier coefficients, a_n and b_n, for $n > 0$ are:

$$\begin{aligned} a_n &= \frac{+1}{n\pi}\left\{\left[\sin(n\theta)|_{\theta=\pi} - \sin(n\theta)|_{\theta=0}\right] - \left[\sin(n\theta)|_{\theta=2\pi} - \sin(n\theta)|_{\theta=\pi}\right]\right\} \\ &= \frac{+1}{n\pi}\left\{\left[\sin(n\pi) - \sin(0)\right] - \left[\sin(2n\pi) - \sin(n\pi)\right]\right\} = \frac{1}{n\pi}\left\{\left[0-0\right] - \left[0-0\right]\right\} = 0 \end{aligned}$$

since $sin(0) = sin(n\pi) = sin(2n\pi) = 0$ for *all* integers, n = 1, 2, 3, 4,..... , and:

$$\begin{aligned} b_n &= \frac{-1}{n\pi}\left\{\left[\cos(n\theta)|_{\theta=\pi} - \cos(n\theta)|_{\theta=0}\right] - \left[\cos(n\theta)|_{\theta=2\pi} - \cos(n\theta)|_{\theta=\pi}\right]\right\} \\ &= \frac{-1}{n\pi}\left\{\left[\cos(n\pi) - \cos(0)\right] - \left[\cos(2n\pi) - \cos(n\pi)\right]\right\} = \frac{-1}{n\pi}\left\{\left[\cos(n\pi) - 1\right] - \left[1 - \cos(n\pi)\right]\right\} \\ &= \frac{-2}{n\pi}\left[\cos(n\pi) - 1\right] = \frac{2}{n\pi}\left[1 - \cos(n\pi)\right] \end{aligned}$$

Now $cos(0) = cos(2n\pi) = +1$ for *<u>all</u>* integers, n = 1, 2, 3, 4,..... , and $cos(n\pi) = +1$ for the *<u>even</u>* integers, n = 2, 4, 6, 8, ..., and $cos(n\pi) = -1$ for the *<u>odd</u>* integers, n = 1, 3, 5, 7, ...

Thus, we see that *<u>all</u>* of the Fourier coefficients, a_n for the *<u>even</u>* functions, $cos(\theta_n)$ vanish i.e. $a_n = 0$ for *<u>all</u>* integers, n = 1, 2, 3, 4, ...

The Fourier coefficients, b_n for the *<u>odd</u>* functions, $sin(\theta_n)$ vanish for the *<u>even</u>* harmonics, i.e. $b_n = 0$ when n = 2, 4, 6, 8, ..., but the Fourier coefficients, b_n are non-zero for the *<u>odd</u>* harmonics, when n = 1, 3, 5, 7, ..., where $b_n = +4/n\pi$.

Thus, the Fourier series expansion of a periodic, bipolar, 50% duty-cycle square wave as shown in the above figure is given by:

$$f(\theta)|_{\substack{square\\-wave}} = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos\theta_n + \sum_{n=1}^{n=\infty} b_n \sin\theta_n = \frac{4}{\pi}\sum_{\substack{n=1\\odd-n}}^{n=\infty} \frac{\sin(n\theta)}{n}$$

Using the replacement: $n_{odd} = 2m - 1$, m = 1, 2, 3, 4, ... in the above summation, we can alternatively write the Fourier series expansion for this square wave as:

$$f(\theta)|_{\substack{square\\-wave}} = \frac{4}{\pi}\sum_{m=1}^{m=\infty} \frac{\sin[(2m-1)\theta]}{(2m-1)} = \frac{4}{\pi}\left\{\sin\theta + \tfrac{1}{3}\sin 3\theta + \tfrac{1}{5}\sin 5\theta + \tfrac{1}{7}\sin 7\theta + ...\right\}$$

Thus, we see that for the periodic, bipolar, 50% duty-cycle square wave, only *odd* harmonics (i.e. *odd* integer multiples of the fundamental) are present in this waveform.

Since the *even* Fourier coefficients, $a_n = 0$ for *all* harmonics ($n \geq 0$) of this waveform, then the *magnitudes* of the Fourier amplitudes, $|r_n|$ associated with the *odd* harmonics are $|r_n| = (a_n^2 + b_n^2)^{½} = b_n = 4/n\pi$ for *odd* n = 1, 3, 5, ... etc., i.e. $|r_{2m-1}| = b_{2m-1} = -4/(2m-1)\pi$ for m = 1, 2, 3, 4, 5... etc. Note that *all* phase angles for these *odd* harmonics are $\delta_n = tan^{-1}(b_n / a_n) = tan^{-1}(\infty) = ½\,\pi = 90^o$ (or equivalently, $\delta_n' = tan^{-1}(a_n / b_n) = tan^{-1}(0) = 0 = 0^o$, since $\delta_n' = \pi/2 - \delta_n = 90^o - \delta_n$).

Only the *odd* harmonics are present in the periodic, bipolar, 50% duty-cycle square wave (as drawn in the figure above) because this waveform, as we have discussed above, has intrinsically *odd* reflection symmetry properties! Thus, simply recognizing the symmetry properties of a waveform instantly tells one which harmonics of the fundamental will or will not be present! Note that the use of symmetry arguments very often is extremely powerful and helpful in terms of gaining insight into the behavior of a physical system!

If we had *flipped* the *polarity* of the waveform, such that initially the waveform was *negative* during the first half of its cycle, then *positive* during the second half of its cycle, this waveform would *still* have odd symmetry, and thus *still* contain the same odd harmonics. However, for this waveform, the *sign* of the non-zero, odd Fourier coefficients, b_n would *reverse* - i.e. $b_n = -4/n\pi$, for odd n = 1, 3, 5, ... etc. One can see this by inspection of the details of working out the above inner production computation for the determination of the b_n Fourier coefficients, as well as from the use of reflection symmetry arguments.

If we had *shifted* the *offset* (e.g. by one unit) of the original periodic, bipolar, 50% duty-cycle square wave, such as to make this waveform a *unipolar* square wave, by adding a d.c. offset (i.e. constant term) to the waveform, then this would only affect the a_0 term in the Fourier series expansion of the waveform. For an upward-shifted unipolar square wave of unit amplitude, for one cycle, the mathematical description of such a wave is given by:

$$f(\theta) = f(\omega t) = +2 \quad \text{for} \quad 0 \leq \theta < \pi$$

and:

$$f(\theta) = f(\omega t) = 0 \quad \text{for} \quad \pi \leq \theta < 2\pi$$

The corresponding $n = 0$ Fourier coefficient for this waveform is $a_0 = 2$. The mean, or average value of $f(\theta)$, averaging over one cycle of this *unipolar*, 50% duty-cycle square wave is $\langle f(\theta)\rangle = 1$ ($= a_0/2 = 2/2 = 1$).

Note that if we had used a periodic, bipolar, 50% duty-cycle square wave which had an amplitude of A_i (instead of a *unit* amplitude), then from the inner product computation of the odd Fourier coefficients, b_n we would instead have obtained $b_n = 4A_i/n\pi$ for the odd harmonics, n = 1, 3, 5, 7, ... etc. Since mathematically, such a waveform would be defined as:

$$f(\theta) = f(\omega t) = +A_i \quad \text{for} \quad 0 \leq \theta < \pi$$

and:

$$f(\theta) = f(\omega t) = -A_i \quad \text{for} \quad \pi \leq \theta < 2\pi$$

If we had shifted the *phase* of the original periodic, bipolar, 50% duty-cycle square wave by e.g. $\Delta\theta = \pm\,\pi/2 = \pm\,90^o$, then this would change the odd-symmetry nature of the waveform to even symmetry. Mathematically, this even-symmetry square wave would be described as:

$$f(\theta) = f(\omega t) = -1 \quad \text{for} \quad 0 \le \theta < \pi/2$$

and:

$$f(\theta) = f(\omega t) = +1 \quad \text{for} \quad \pi/2 \le \theta < 3\pi/2$$

and:

$$f(\theta) = f(\omega t) = -1 \quad \text{for} \quad 3\pi/2 \le \theta < 2\pi$$

The Fourier coefficients for this even-symmetry waveform would be $a_0 = 2$, *all* even-n Fourier coefficients, $a_n = 0$, for $n = 2, 4, 6$, ... etc., but *all* odd-n Fourier coefficients, $a_n = 4/n\pi$ for $n = 1, 3, 5, 7$, ... etc. and *all* Fourier coefficients, $b_n = 0$ for *all* $n = 1, 2, 3, 4, 5, 6$, ... etc.

If we had shifted the phase of the original periodic, bipolar, 50% duty-cycle square wave by e.g. a random shift, $\Delta\theta$, then very likely the resulting waveform would have <u>*neither*</u> odd <u>*nor*</u> even reflection symmetry properties (unless $\Delta\theta$ happened to be e.g. $\pm\,\pi/2$ or $\pm\,\pi$). Mathematically, this waveform would be described as:

$$f(\theta) = f(\omega t) = -1 \quad \text{for} \quad 0 \le \theta < \Delta\theta$$

and:

$$f(\theta) = f(\omega t) = +1 \quad \text{for} \quad \Delta\theta \le \theta < \Delta\theta + \pi$$

and:

$$f(\theta) = f(\omega t) = -1 \quad \text{for} \quad \Delta\theta + \pi \le \theta < 2\pi$$

For a waveform that has *no* reflection symmetry properties whatsoever, in general *all* of the Fourier coefficients, a_0, the a_n and b_n coefficients will be non-zero.

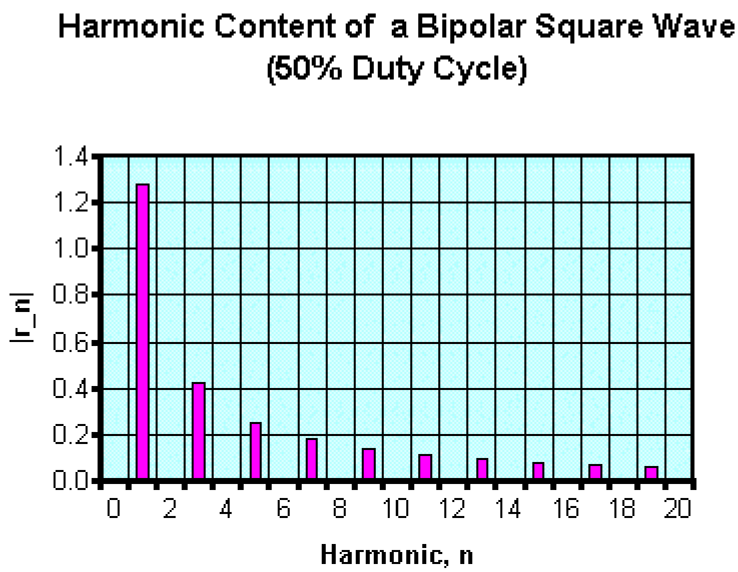
<u>Because</u> of the existence of reflection symmetries in a waveform, certain of the Fourier coefficients, a_0, the a_n and/or b_n will <u>vanish</u>.

However, even for "no-symmetry" waveforms, as we have discussed above, for each harmonic, n of the fundamental, there is physically only *one* amplitude, $|r_n| = (a_n^2 + b_n^2)^{½}$ and *one* phase angle, $\delta_n = \tan^{-1}(b_n / a_n)$ (or equivalently $\delta_n' = \tan^{-1}(a_n / b_n)$) associated with that harmonic.

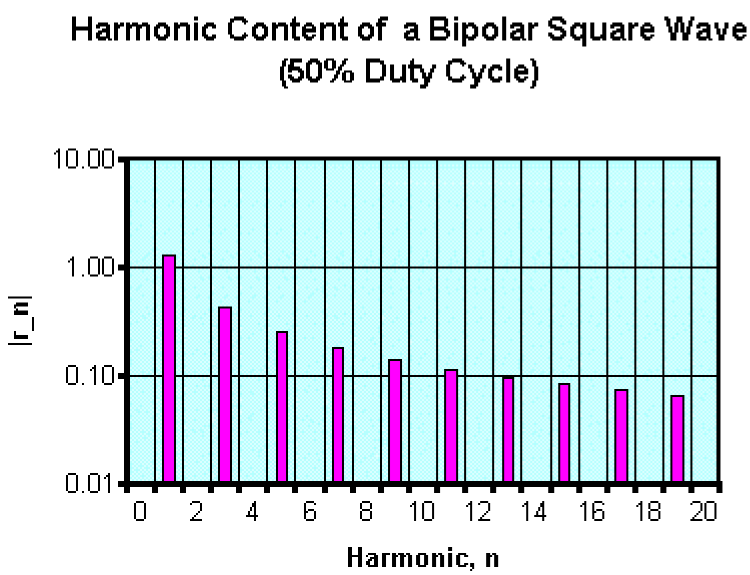
If the <u>*duty-cycle*</u> of the waveform is varied from its "symmetrical" value of 50%, this will have a corresponding impact on *all* of the Fourier coefficients, a_0, the a_n and/or b_n, e.g. since the d.c. value of e.g. a 10% duty-cycle bipolar square wave certainly is not 0! We will discuss this case further, below.

For values of the duty cycle other than "easy" choices of integer fractions of the full cycle in the "generic" theta variable (i.e. $\Delta\theta = \theta_2 - \theta_1 = 2\pi$), the evaluation of the inner products used to determine the Fourier coefficients, a_0, the a_n and b_n can be *very* tedious to carry out by hand. However, these *are* straightforward to carry out on a computer, using e.g. numerical integration techniques.

The amplitudes, $|r_n| = (a_n^2 + b_n^2)^{½} = 4/n\pi$ for the first twenty harmonics (i.e. $n < 20$) associated with the periodic, bipolar, 50% duty-cycle, unit amplitude square wave are shown in the figure below:



The following figure shows the same information as above, except that it is shown as a *semi-log* plot:



As can be seen from the above figures, in addition to the fundamental, at frequency, *f*, only the *odd* harmonics, at frequencies 3*f*, 5*f*, 7*f*, 9*f*, ... etc. contribute to creating this waveform.

Note that the *ratio* of harmonic amplitudes for this square wave, relative to the fundamental is $|r_n| / |r_1| = 1/n$, which does not decrease very fast, as the harmonic #, n increases (n = 3, 5, 7, 9, ... etc.).

As far as harmonic content goes, *any* kind of square wave, compared to just about any other kind of waveform is *extremely* rich in harmonics. The reason for this is due to the very sharp "breaks" or "jumps" (i.e. the discontinuities) in the waveform. To *make* such sharp edges in the waveform, extremely high harmonics, with correspondingly very short wavelengths are needed, even though their relative amplitudes may be small.

The human ear hears a square-wave audio signal as being very "bright", relative to e.g. a pure-tone (sine-wave) audio signal at the same frequency, which sounds "mellow" or "round", since it has only a single harmonic component - the fundamental. In fact, the square wave audio signal also sounds "harsh" to the human ear, because of the presence of all of the *odd* harmonics, at 3*f*, 5*f*, 7*f*, 9*f*, ... etc.

Note that harmonics at the frequencies 3*f*, 5*f*, 7*f*, 9*f*, ... etc. are not integer-multiples of an octave above the fundamental, at frequency, *f*. A frequency that is one octave above the fundamental is at 2*f*; two octaves above, at 4*f*; three octaves above, at 6*f*, ... etc.

If the fundamental is at a frequency, f = 440 *Hz* (i.e. A_4 on a piano), then 3f = 1320 *Hz* (very close to E_6 on a piano), which is one octave and a *fifth* above the fundamental. The fifth harmonic is 5f = 2200 *Hz* (very close to $C_7^{\#}$ on a piano), which is two octaves and a *third* above the fundamental. Together, ignoring the octaves, these two harmonics, in combination with the fundamental, form a major triad-type chord (in the key of *A*, here), so it isn't *that* displeasing to the human ear to listen to a square wave-type of sound.

If a square wave signal, e.g. created by a function generator is output through a loudspeaker, converting it to sound, the human ear perceives the *loudness*, *L* of this sound (units of *deci-Bels*, abbreviated as *dB*) which is *logarithmically* proportional to the *intensity*, *I*) of the sound wave (units of $Watts/m^2$), which in turn is linearly proportional power, *P* of the sound wave (units of *Watts*), which in turn is proportional to the *square* of the amplitude, A_i of the square wave. Mathematically:

$$Loudness,\ L \equiv 10 \log_{10}(I / I_o) \quad \text{(units = } deci\text{-}Bels,\ dB)$$

$$Intensity,\ I\ (Watts/m^2) \propto Power,\ P\ (Watts) \propto \{Output\ Response,\ R_o(S_i(t))\}^2$$

The *threshold* of human hearing - i.e. the faintest possible sound that is detectable as such by the (average) human ear is defined as *Loudness*, $L_o \equiv 0\ dB$, which corresponds to a sound intensity, I_o associated with the threshold of human hearing of $I_o = 10^{-12}\ Watts/m^2$.

If the loudness of the fundamental (n = 1) is $L_1 = 60\ dB\ (100\ dB)$, this corresponds to an intensity associated with the fundamental tone of $I_1 = 10^{-6}\ (10^{-2})\ Watts/m^2$, respectively. If the *ratio* of the amplitude for the n^{th} harmonic to the amplitude of the fundamental associated with the square wave is $|r_n| / |r_1| = 1/n$, for *odd* n = 3, 5, 7, 9, ... etc. Then the ratio of intensity for the n^{th} harmonic to the intensity for the fundamental associated with the square wave is $I_n / I_1 = (1/n)^2$, and the terms, e.g for n = 3 are:

$$log_{10}(I_n / I_1) = log_{10}(1/n)^2 = 2\ log_{10}(1/n) = 2\ log_{10}(0.3333) = -0.9542$$

and

$$log_{10}(I_1 / I_o) = 6\ (10) \quad \text{for} \quad I_1 = 10^{-6}\ (10^{-2})\ Watts/m^2, \quad \text{respectively.}$$

Thus, the human ear will perceive the loudness, L_n of the n^{th} harmonic, relative to perceived loudness, L_1 of the fundamental of the square wave, as heard e.g. through a loudspeaker as:

$$L_n / L_1 = 10\ log_{10}(I_n / I_o) / 10\ log_{10}(I_1 / I_o) = log_{10}(I_n / I_o) / log_{10}(I_1 / I_o)$$
$$= log_{10}[(I_n / I_1)*(I_1 / I_o)] / log_{10}(I_1 / I_o)$$
$$= [log_{10}(I_n / I_1) + log_{10}(I_1 / I_o)] / log_{10}(I_1 / I_o)$$
$$= \{log_{10}(I_n / I_1) / log_{10}(I_1 / I_o)\} + 1$$
$$= 1 + \{log_{10}(I_n / I_1) / log_{10}(I_1 / I_o)\}$$

Then for the 3^{rd} harmonic:

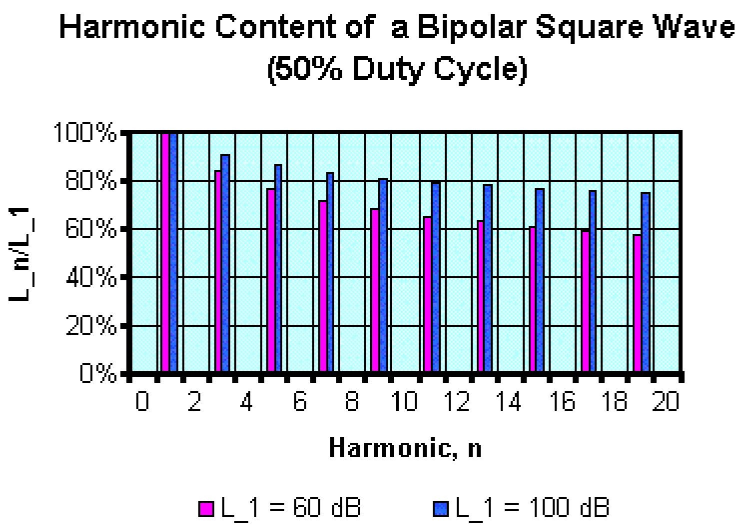
$$L_3 / L_1 = 1 - \{0.9542 / 6\} \quad (= 1 - \{0.9542 / 10\})$$
$$= 84.1\% \qquad (= 90.5\%)$$

for $I_1 = 10^{-6}\ (10^{-2})\ Watts/m^2$, respectively. This is the (fractional) amount of third harmonic, as heard by the human ear for a square wave. This is very large! Note also that the ratio, L_n / L_1 increases (logarithmically) with increasing amplitude of the square wave! For a loudness of the fundamental tone of $L_1 = 60\ dB\ (100\ dB)$, the loudness of the third harmonic, for $|r_3| / |r_1| = 1/3 = 33.3\%$ is:

$$L_3 = 10\ log_{10}(I_3 / I_o) = 10\ log_{10}[(I_3 / I_1)*(I_1 / I_o)]$$
$$= 10\ log_{10}(I_3 / I_1) + 10\ log_{10}(I_1 / I_o)$$
$$= 20\ log_{10}(0.3333) + 60\ dB\ (100\ dB)$$
$$= -9.54\ dB + 60\ dB\ (100\ dB)$$
$$= 50.46\ dB\ (90.46\ dB), \text{ respectively.}$$

Thus, a value of the ratio of amplitudes for the third harmonic to the fundamental of the square wave, $|r_3| / |r_1| = 1/3 = 33.3\%$ is perceived by the human ear as *extremely* "rich" in third harmonic content. The human ear is capable of detecting quite small harmonic overtone components, where $|r_n| / |r_1| \sim 0.5\%$ (or less) because these (still) correspond to large values of the ratio $L_n / L_1 \sim 25\%$ ($\sim 55\%$), for $L_1 = 60\ dB\ (100\ dB)$, respectively!

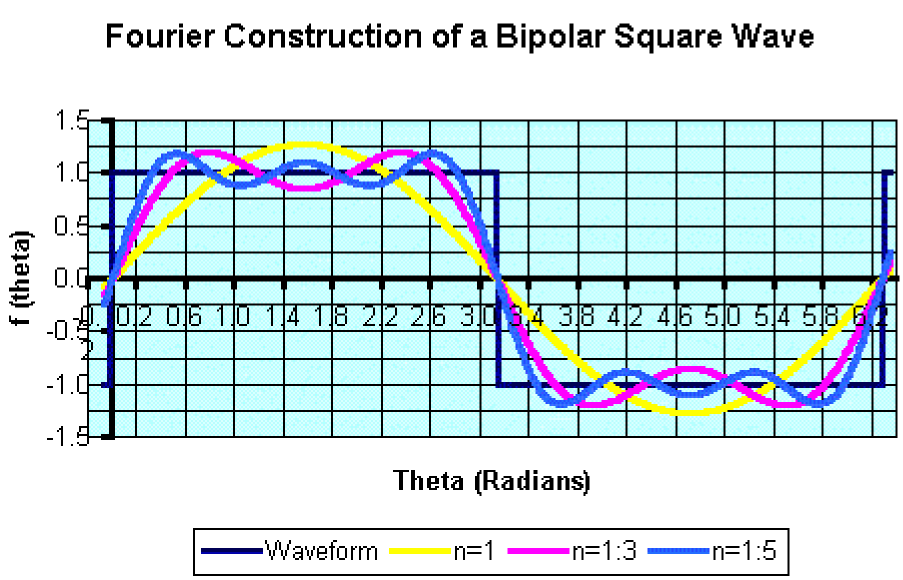
The following figure shows the loudness ratios, L_n / L_1 for the first twenty harmonics (i.e. $n < 20$) associated with the bipolar square wave, for loudness values of the fundamental of $L_1 = 60\ dB$ (~ quiet) and for $L_1 = 100\ dB$ (~ quite loud). This is what the human ear perceives as the loudness of the harmonics relative to that of the fundamental. Note that the decrease in the loudness ratio, L_n / L_1 with increasing harmonic #, n is quite slow.



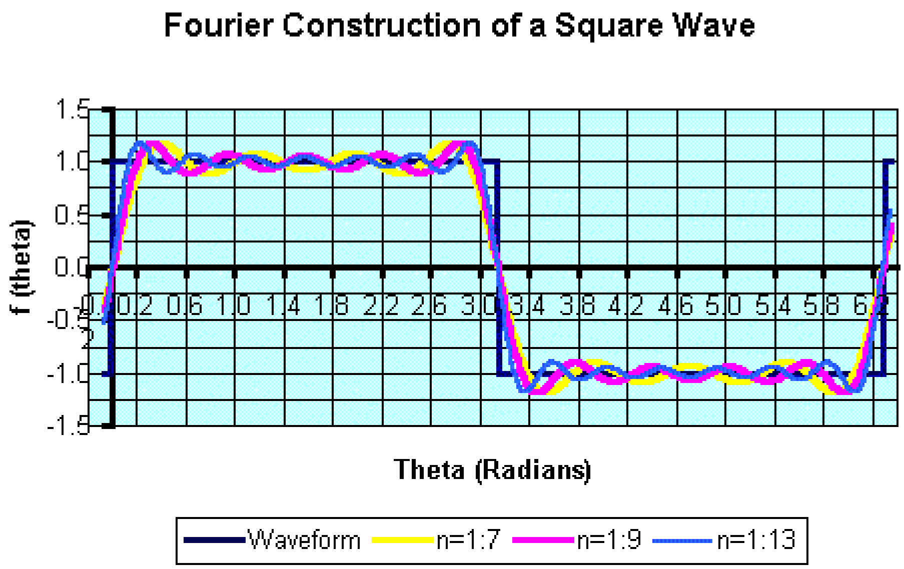
The following two figures show the "Fourier construction" of a periodic, bipolar, 50% duty-cycle unit-amplitude square wave. The waveforms in these figures were generated using truncated, finite-term version(s) of the Fourier series expansion for this waveform:

$$f(\theta)\,|_{\substack{square\\-wave}} = \frac{4}{\pi}\sum_{m=1}^{m=\infty}\frac{\sin[(2m-1)\theta]}{(2m-1)} = \frac{4}{\pi}\left\{\sin\theta + \tfrac{1}{3}\sin 3\theta + \tfrac{1}{5}\sin 5\theta + \tfrac{1}{7}\sin 7\theta + ...\right\}$$

The first figure shows the bipolar square wave (labelled as "Waveform") overlaid with three other waveforms: that associated with just the fundamental ("n = 1"), then the waveform associated with fundamental + 3rd harmonic ("n = 1:3"), then the waveform associated with fundamental + 3rd + 5th harmonic ("n = 1:5").



The second figure shows the bipolar square wave (labelled as “Waveform”) overlaid with three other waveforms: that associated with the fundamental through the 7*th* harmonic (“*n* = 1:7”), then the waveform associated with fundamental through the 9*th* harmonic (“*n* = 1:9”), then the waveform associated with fundamental through the 13*th* harmonic (“*n* = 1:13”).



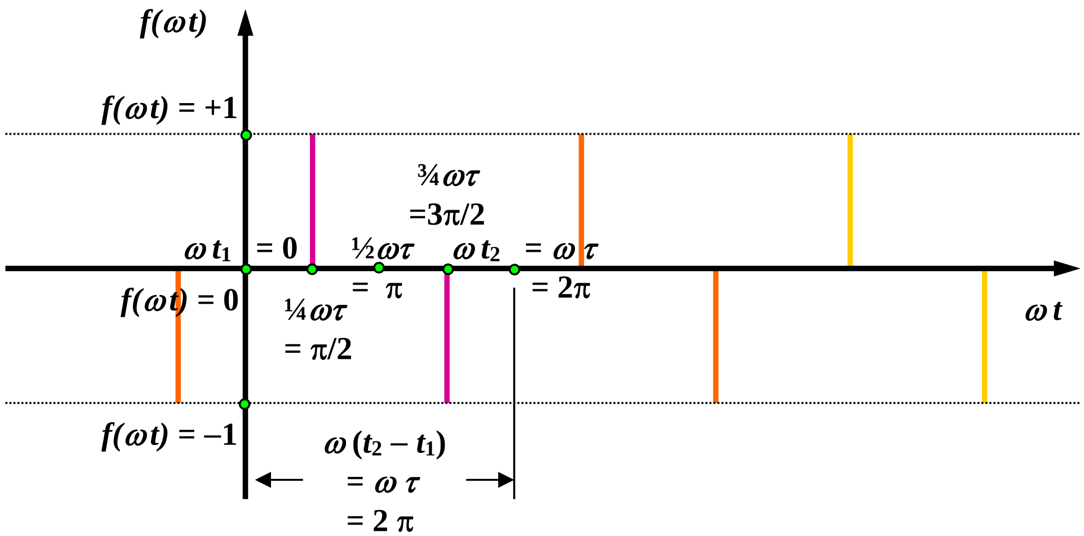
As each of the higher harmonic terms is added in, “building” the Fourier series for the bipolar square wave, the agreement between each successive waveform and that of the actual bipolar square wave becomes better and better. As stater earlier, the higher harmonic terms are required to achieve good agreement in the most rapidly-changing portions of this waveform, as can be seen from these two figures.

In rock music, the square wave shows up on the output side of various kinds of “fuzz” (i.e. distortion) effect (FX) “stomp” boxes used for altering the signal(s) from electric guitars, most notably used e.g. by heavy-metal bands. In many of these type of distortion FX boxes, the signal gain is very high. Somewhere in the FX box circuit, a non-linear circuit element, such as a pair of back-to-back diodes “clips” the large-amplitude signal from the guitar, chopping off (i.e. limiting) the peaks, thus creating a square wave-type signal. Note also that the FX box *also* additionally acts as a *signal compressor/limiter*, as a consequence of clipping the large-amplitude input waveform. Another interesting aspect of the use of distortion FX boxes is that the resulting high-harmonic content sound wave output from the guitar amplifier can acoustically couple back to the strings of the electric guitar, providing the necessary energy to drive the strings into so-called “infinite-sustain”, also known as feedback. This acoustical feedback coupling is (usually) *not* via the fundamental; it occurs *primarily* through the acoustical feedback coupling associated with the 3*rd* harmonic of the square wave!

Square waves also have use(s) in electronic keyboard-type instruments, as part of a large “pallette” of keyboard sounds.

C. Fourier Analysis of a Periodic, Bipolar Delta-Function

The limiting case of the duty cycle going to zero for a temporally-periodic, bipolar square wave of is known as a periodic, bipolar *delta-function waveform*, consisting of a series of alternating up and down "spikes", each of zero width, as shown in the figure below:



Such "spikes" can be represented mathematically by a so-called *delta-function*, $\delta(x)$. The mathematical properties of the delta-function, $\delta(x)$ are quite intriguing. The delta-function, $\delta(x)$ is located at the position of its argument, here, $x = 0$. Thus, e.g. $\delta(x - x_o)$ is located at $x = x_o$, and thus $\delta(x + x_o)$ is located at $x = - x_o$. (n.b. The argument, u of the delta-function, $\delta(u)$ is *always* equal to zero, e.g. $u = (x - x_o) = 0$, thus $x = x_o$).

Formally, mathematically, the delta-function, $\delta(x)$ has *zero* width and *infinite* height, but *only* at $x = 0$. It is *zero* everywhere else. When a delta-function is used inside of an integral, amazing things happen as a result. For example, if the range of integration contains the point $x = x_o$, then:

$$\int \delta(x - x_o)dx = 1 \qquad\qquad \int f(x)\delta(x - x_o)dx = f(x_o)$$

otherwise both of these integrals are = 0, if $x = x_o$ is *not* contained within the range of integration. Note also that the (one-dimensional) delta-function, $\delta(x)$ has dimensions (i.e. units) of $1/x$.

Mathematically, we can define the above *odd-symmetry* waveform, $f(\theta)$ over the interval $0 \le \theta < 2\pi$ (i.e. one cycle of this waveform) as:

$$f(\theta) = \delta(\theta - \pi/2) - \delta(\theta - 3\pi/2)$$

Thus, the positive-going delta-function is located at $\theta = \frac{1}{4}\omega\tau = \pi/2$, and the negative-going delta-function is located at $\theta = \frac{3}{4}\omega\tau = 3\pi/2$.

We can then determine the Fourier coefficients, a_0, the a_n and b_n from their associated inner products:

$$a_0 = \frac{1}{\pi}\langle f(\theta),1\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)d\theta = \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi} f(\theta)d\theta + \int_{\theta=\pi}^{\theta=2\pi} f(\theta)d\theta\right]$$
$$= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi} \delta(\theta-\tfrac{\pi}{2})d\theta - \int_{\theta=\pi}^{\theta=2\pi} \delta(\theta-\tfrac{3\pi}{2})d\theta\right] = \frac{1}{\pi}[1-1] = 0$$

The Fourier coefficients, a_n and b_n for $n > 0$ are:

$$a_n = \frac{1}{\pi}\langle f(\theta),\cos(\theta_n)\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)\cos(\theta_n)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)\cos(n\theta)d\theta$$
$$= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi} f(\theta)\cos(n\theta)d\theta + \int_{\theta=\pi}^{\theta=2\pi} f(\theta)\cos(n\theta)d\theta\right]$$
$$= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi} \delta(\theta-\tfrac{\pi}{2})\cos(n\theta)d\theta - \int_{\theta=\pi}^{\theta=2\pi} \delta(\theta-\tfrac{3\pi}{2})\cos(n\theta)d\theta\right] = \frac{1}{\pi}\left[\cos(\tfrac{n\pi}{2}) - \cos(\tfrac{3n\pi}{2})\right]$$

$$b_n = \frac{1}{\pi}\langle f(\theta),\sin(\theta_n)\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)\sin(\theta_n)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)\sin(n\theta)d\theta$$
$$= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi} f(\theta)\sin(n\theta)d\theta + \int_{\theta=\pi}^{\theta=2\pi} f(\theta)\sin(n\theta)d\theta\right]$$
$$= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi} \delta(\theta-\tfrac{\pi}{2})\sin(n\theta)d\theta - \int_{\theta=\pi}^{\theta=2\pi} \delta(\theta-\tfrac{3\pi}{2})\sin(n\theta)d\theta\right] = \frac{1}{\pi}\left[\sin(\tfrac{n\pi}{2}) - \sin(\tfrac{3n\pi}{2})\right]$$

Now:

$\cos(n\pi/2) = \cos(3n\pi/2) = 0$ for *odd* $n = 1, 3, 5, 7, \ldots$ etc.

However:

$\cos(n\pi/2) = \cos(3n\pi/2) = -1$ for *even* $n = 2, 6, 10, 14, \ldots$ etc.

but:

$\cos(n\pi/2) = \cos(3n\pi/2) = +1$ for *even* $n = 4, 8, 12, 16, \ldots$ etc.

Thus, for *all* integers, $n > 0$, the Fourier coefficients, $a_n = 0$.

Now:

$\sin(n\pi/2) = -\sin(3n\pi/2) = +1$ for *odd* $n = 1, 5, 9, 13, \ldots$ etc.

but:

$\sin(n\pi/2) = -\sin(3n\pi/2) = -1$ for *odd* $n = 3, 7, 11, 15, \ldots$ etc.

However:

$\sin(n\pi/2) = \sin(3n\pi/2) = 0$ for *even* $n = 2, 4, 6, 8, \ldots$ etc.

Thus, only the *odd* Fourier coefficients, b_n are non-zero:

For *odd* $n = 1, 5, 9, 13, \ldots$ etc., $b_n = 2/\pi$. For *odd* $n = 3, 7, 11, 15, \ldots$ etc., $b_n = -2/\pi$.

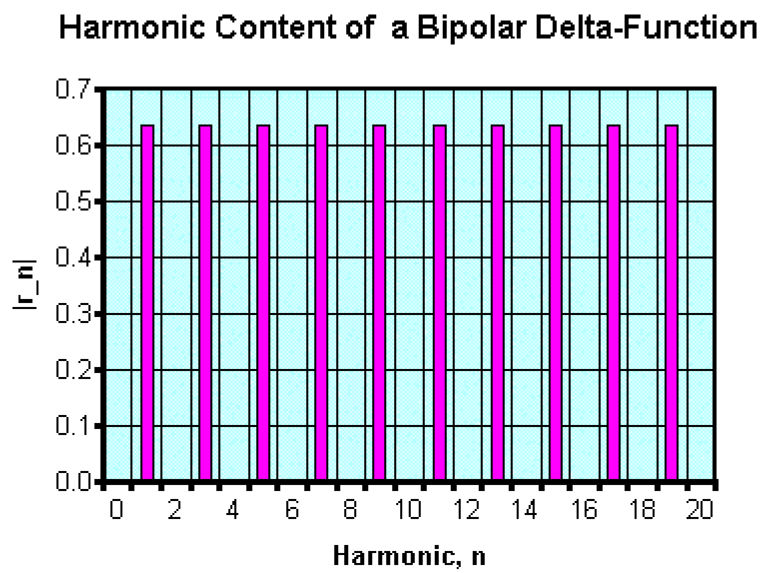
Thus, the Fourier series expansion for a periodic, bipolar, delta-function wave as shown in the above figure is given by:

$$f(\theta)\,|_{\substack{\delta-fcn \\ -wave}} = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos\theta_n + \sum_{n=1}^{n=\infty} b_n \sin\theta_n = \frac{2}{\pi} \sum_{\substack{n=1 \\ odd-n}}^{n=\infty} (-1)^{(n-1)/2} \sin(n\theta)$$

Using the replacement: $n_{odd} = 2\,m - 1$, m = 1, 2, 3, 4, ... in the above summation, we can alternatively write the Fourier series expansion for this delta-function wave as:

$$f(\theta)\,|_{\substack{\delta-fcn \\ -wave}} = \frac{2}{\pi} \sum_{m=1}^{m=\infty} (-1)^{m-1} \sin[(2m-1)\theta] = \frac{2}{\pi} \left\{ \sin\theta - \sin 3\theta + \sin 5\theta - \sin 7\theta + \sin 9\theta - ..... \right\}$$

Note that the *magnitudes* of the non-zero amplitudes of the harmonics, $|r_n| = |b_n| = 2/\pi$, as shown in the figure below for the first 20 harmonics.



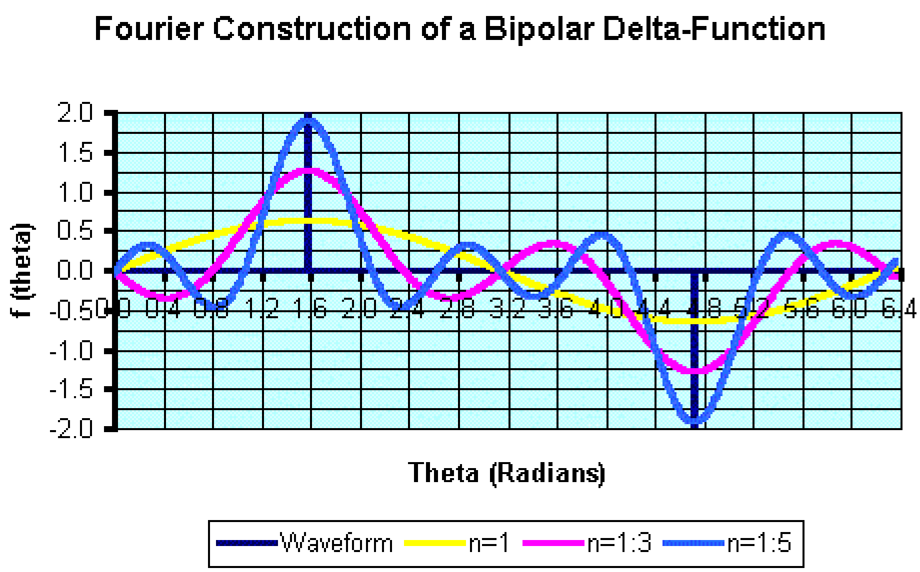
Note that the $|r_n|$ have no n-dependence - i.e. they are independent of frequency! Thus for a bipolar delta-function waveform, <u>*all*</u> *odd* harmonics contribute <u>equally</u> in magnitude to creating this waveform!

However, because the *sign* of the b_n changes with successive *odd* integer, n, this also means that the phase angle, δ_n changes sign with successive *odd* integer, n. For *odd* n = 1, 5, 9, 13, ... etc., where $b_n = +2/\pi$ and $a_n = 0$, then $\delta_n = \tan^{-1}(b_n / a_n) = \tan^{-1}(\infty) = ½\,\pi = 90^o$. For *odd* n = 3, 7, 11, 15, ... etc., where $b_n = -2/\pi$ and $a_n = 0$, then $\delta_n = \tan^{-1}(b_n / a_n) = \tan^{-1}(-\infty) = -½\,\pi = -90^o$. Thus, the non-zero, successive odd-n phase angles, δ_n of the harmonics are 180^o degrees apart - i.e. successive harmonics tend to *cancel* against each other, *except* in the regions $\theta \sim \pi/2$ and $\theta \sim 3\pi/2$!

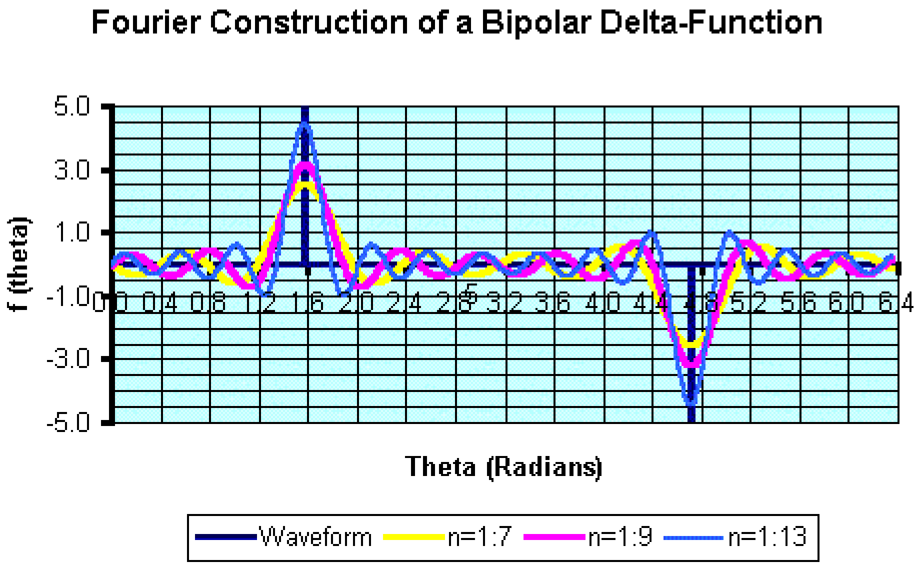
The following two figures show the "Fourier construction" of a periodic, bipolar, 50% duty-cycle unit-amplitude delta-function wave. The waveforms in these figures were generated using truncated, finite-term version(s) of the Fourier series expansion for this waveform:

$$f(\theta)|_{\substack{\delta-fcn \\ -wave}} = \frac{2}{\pi}\sum_{m=1}^{m=\infty}(-1)^{m-1}\sin[(2m-1)\theta] = \frac{2}{\pi}\{\sin\theta - \sin 3\theta + \sin 5\theta - \sin 7\theta + \sin 9\theta - .....\}$$

The first figure shows the bipolar delta-function wave (labelled as "Waveform") overlaid with three other waveforms: that associated with just the fundamental ("*n* = 1"), then the waveform associated with fundamental + 3rd harmonic ("*n* = 1:3"), then the waveform associated with fundamental + 3rd + 5th harmonic ("*n* = 1:5").



The second figure shows the bipolar delta-function wave (labelled as "Waveform") overlaid with three other waveforms: that associated with the fundamental through the 7th harmonic ("*n* = 1:7"), then the waveform associated with fundamental through the 9th harmonic ("*n* = 1:9"), then the waveform associated with fundamental through the 13th harmonic ("*n* = 1:13").



It can be seen that the higher-order harmonics are much needed for decreasing the *width* of the "pulse" at each delta-function location. The width of each pulse slowly decreases as the number of harmonics included in the "Fourier construction" of the bipolar delta function increases. Only in the limit of using an *infinite* number of harmonics does the width of each delta-function "pulse" formally become zero.

The periodic, bipolar delta-function waveform is the "0% duty-cycle" limiting case of the periodic, bipolar 50% duty-cycle square wave. While the harmonic content of the 50% duty-cycle bipolar square wave is already extremely rich in *odd* harmonics, with harmonic amplitudes that decrease slowly, as $1/n$ of the harmonic #, n, the "0% duty-cycle" delta-function waveform is *<u>the</u> extreme* in harmonic content, since all harmonics have the *<u>same</u>* amplitude. For bipolar square waves with duty-cycle (DC) between 0% < DC < 50%, the decrease in harmonic content with increasing harmonic # is less steep than $1/n$, becoming *flatter* with increasing harmonic # as the duty cycle decreases from 50%, to the limiting case for DC = 0%, when the harmonic content with increasing harmonic # is perfectly flat.

If the duty-cycle of the periodic, bipolar square wave increases *beyond* 50%, then the only way this can occur is if the waveform develops a d.c. offset. Thus, the Fourier series for such waveforms develops a non-zero value of a_0 (i.e. $|r_0|$) for DC > 50%. For the limiting case of a bipolar, *<u>unit-amplitude</u>* square wave with duty factor, DC = 100%, then the time average of this waveform, $< f(\theta) > = 1 = a_0 / 2$, thus $a_0 = 2$ here. Note that this 100% duty-cycle waveform is also a periodic, but unipolar (i.e. *single*) delta-function waveform, for *each* cycle of the waveform. The 100% duty cycle, unit-amplitude periodic waveform can thus be thought of as a superposition (i.e. linear combination) of a d.c. offset (of strength $a_0 = 2$) with a periodic, unipolar delta-function waveform. Thus, this waveform will also have a perfectly flat harmonic spectrum, neglecting the zero-frequency d.c. offset term.

Exercises:

1. Compute the Fourier coefficients, a_0, a_n and b_n for the "flipped" bipolar, 50% duty-cycle square wave, in the time domain:
$$f(\theta) = f(\omega t) = -1 \quad \text{for} \quad 0 \le \theta < \pi$$
$$f(\theta) = f(\omega t) = +1 \quad \text{for} \quad \pi \le \theta < 2\pi$$
Compare these Fourier coefficients with those obtained above for the "unflipped" bipolar, 50% duty-cycle square wave.

2. Compute the Fourier coefficients, a_0, a_n and b_n for the "shifted" bipolar, 50% duty-cycle square wave, in the time domain:
$$f(\theta) = f(\omega t) = -1 \quad \text{for} \quad 0 \le \theta < \pi/2$$
$$f(\theta) = f(\omega t) = +1 \quad \text{for} \quad \pi/2 \le \theta < 3\pi/2$$
$$f(\theta) = f(\omega t) = -1 \quad \text{for} \quad 3\pi/2 \le \theta < 2\pi$$
Compare these Fourier coefficients with those obtained above for the "unflipped" and "flipped" bipolar, 50% duty-cycle square waves.

3. Compute the Fourier coefficients, a_0, a_n and b_n for the ***unipolar***, 25% duty-cycle square wave, in the time domain:
$$f(\theta) = f(\omega t) = 0 \quad \text{for} \quad 0 \le \theta < \pi/2$$
$$f(\theta) = f(\omega t) = +1 \quad \text{for} \quad \pi/2 \le \theta < 2\pi$$
Compare these Fourier coefficients with those obtained above for the "unflipped" bipolar, 50% duty-cycle square wave.

4. Compute the Fourier coefficients, a_0, a_n and b_n for the ***unipolar*** delta-function waveform, in the time domain:
$$f(\theta) = f(\omega t) = \delta(\theta - \pi)$$
Compare these Fourier coefficients with those obtained above for the bipolar delta-function waveform.

5. For each of the above exercises, use e.g. *MathLab*, or a spreadsheet program, such as *Excel* to make plots of the harmonic amplitudes, $|r_n|$, the loudness ratios, L_n / L_1 and Fourier contruction of the original waveform, for e.g. the first few harmonics.

References for Fourier Analysis and Further Reading:

1. Fourier Series and Boundary Value Problems, 2nd Edition, Ruel V. Churchill, McGraw-Hill Book Company, 1969.

2. Mathematics of Classical and Quantum Physics, Volumes 1 & 2, Frederick W. Byron, Jr. and Robert W. Fuller, Addison-Wesley Publishing Company, 1969.

3. Mathematical Methods of Physics, 2nd Edition, Jon Matthews and R.L. Walker, W.A. Benjamin, Inc., 1964.

Legal Disclaimer and Copyright Notice:

Legal Disclaimer:

The author specifically disclaims legal responsibility for any loss of profit, or any consequential, incidental, and/or other damages resulting from the mis-use of information contained in this document. The author has made every effort possible to ensure that the information contained in this document is factually and technically accurate and correct.

Copyright Notice:

The contents of this document are protected under both United States of America and International Copyright Laws. No portion of this document may be reproduced in any manner for commercial use without prior *written* permission from the author of this document. The author grants permission for the use of information contained in this document for *private*, *non-commercial* purposes only.